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Decomposition of Mathematical Programming Problems by Dynamic Programming and Its Application to Block-Diagonal Geometric Programs

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1. INTRODUCTION

In recent years, decomposition techniques of large-scale mathematical programming problems have been investigated by many authors [3, 7, 8, 11, 13, 18 *et al.*]. The origin of these techniques is the Dantzig-Wolfe decomposition principle [5] for linear programming problems. It deals with problems involving a number of linear subproblems tied together by coupling linear constraints. These techniques also deal with problems of the similar structure involving convex or nonlinear subproblems. Most of those problems are additively separable and called separable programs. The problem discussed in the present paper is a generalized form of the separable program, which will be described exactly in the next section and called the principal problem. The main purpose of the present paper is to find the sufficient condition for the principal problem to be decomposed into subproblems by dynamic programming. The heart of dynamic programming is, as well known, principle of optimality [1, p. 83], which leads to recursive relations. Conversely, the deduced recursive relations imply that principle of optimality is applicable. For a multi-stage decision process, Mitten [12] and Nemhauser [15] have shown the sufficient conditions of a return function that principle of optimality be applicable. Their results, however, are not directly adequate for the principal problem. Because they dealt with stage transforms in stead of constraints. Neuner and Miller [16] discussed the problem similar to the principal problem. The results obtained in the present paper, however, are more general than their results. The second purpose of the present paper is to investigate the principal problem whose subproblems are equivalent to linear programming problems. A block-diagonal form of a geometric program which is called briefly a block-diagonal geometric program is discussed as a typical example. In particular case of the block-diagonal geometric program, the present method of combining parametric linear programming with dynamic programming is of practical use. Numerical examples are given.

2. General Results

Let x_n ($n = 1, 2,..., N$) be vectors $(x_{n1}, x_{n2},..., x_{ni_n})$ on a subspace X_n of an i_n-dimensional Euclidian space E^{i_n}. Let $f_n(x_n)$ and $g_{nm}(x_n)$ be ($m = 1, 2,..., M_n$, $n = 1, 2,..., N$) real valued functions defined on X_n. Let $R(\cdot)$ be the range of the function $\cdot$. The problem to be discussed is:

$$\text{OPT } F(f_N(x_N), f_{N-1}(x_{N-1}),..., f_1(x_1)) \tag{2.1}$$

subject to

$$G_m(g_{Nm}(x_N), g_{N-1m}(x_{N-1}),..., g_{1m}(x_1)) \leqslant 0, \qquad m = 1, 2,..., L, \tag{2.2}$$

$$G_m(g_{Nm}(x_N), g_{N-1m}(x_{N-1}),..., g_{1m}(x_1)) = 0, \qquad m = L + 1,..., M, \tag{2.3}$$

and

$$g_{nm}(x_n) \leqslant 0, \qquad m = M + 1,..., M_n, \qquad n = 1, 2,..., N,$$

where OPT means min or max according to the problem to be solved [10], $F(\cdot)$ and $G_m(\cdot)$ are real valued functions defined on subspaces of E^N including $\mathsf{X}_{n-1}^N R(f_n)$ and $\mathsf{X}_{n=1}^N R(g_{nm})$, respectively. Without loss of generality, it can be assumed that

$$X_n = \{x_n ; g_{nm}(x_n) \leqslant 0,\ m = M + 1,..., M_n\}.$$

The above problem is hereafter refered to as the *principal problem* and abbreviated to the following:

$$\begin{aligned} P \equiv \text{OPT}\{F(f_N,..., f_1) \mid\ & G_m(g_{Nm},..., g_{1m}) \leqslant 0\ (m = 1,..., L), \\ & G_m(g_{Nm},..., g_{1m}) = 0\ (m = L + 1,..., M) \text{ and } x_n \in X_n \\ & (n = 1,..., N)\}. \end{aligned}$$

The objective function (2.1) is called the *principal objective function* and the functions of the first constraint (2.2) and the equality constraint (2.3) are called the *principal constraint functions* and the *principal equality constraint functions*, respectively. It should be noted that the principal problem is a generalized form of a separable program. The main purpose of this section is to find the sufficient conditions of the principal objective function, the principal constraint functions and the principal equality constraint functions for the principal problem to be decomposed into subproblems by dynamic programming. Therefore, under those conditions (if exist), the principal problem can be solved by composing iteratively solutions of subproblems by dynamic programming.

Let y and z with or without subscript be real number, $H(y_N, y_{N-1}, \dots, y_1)$ a real valued function defined on a subset of E^N, and $h_n(y, z)$ ($n = 2, 3, \dots, N$) real valued functions defined on subsets of E^2. The real valued function H is said to be *separable*, if and only if there exist real valued functions h_n ($n = 2, 3, \dots, N$) such that

$$H(y_N, y_{N-1}, \dots, y_1) = h_N(y_N, h_{N-1}(y_{N-1}, \dots, h_2(y_2, y_1)^{N-1\text{-tuple}} \cdots)).$$

That is, if separable, H is represented as follows:

$$H(y_N, y_{N-1}, \dots, y_1) = H_N(y_N, y_{N-1}, \dots, y_1), \tag{2.4}$$

where $H_n (n = 1, 2, \dots, N)$ are iteratively defined as

$$H_1(y_1) = y_1 \tag{2.5}$$

and for $n = 2, 3, \dots, N$,

$$H_n(y_n, y_{n-1}, \dots, y_1) = h_n(y_n, H_{n-1}(y_{n-1}, \dots, y_1)). \tag{2.6}$$

The above functions h_n are called the *separating functions* of H. The real valued function H is said to be *weakly decomposable* (*strictly decomposable* or *left-continuous*), if H is separable and all separating functions $h_n(y, z)$ are nondecreasing (increasing or left-continuous) with respect to z. Moreover, the real valued function H is said to be *strongly decomposable*, if H is separable and all separating functions $h_n(y, z)$ are nondecreasing with respect to both y and z. It is to be noted that weak decomposability is identical with the condition shown by Nemhauser [15, p. 35].

To begin with, suppose that the principal constraint function $G_m (m \leqslant L)$ is weakly decomposable and left-continuous with separating functions h_{nm} ($n = 2, 3, \dots, N$). Let $V_{nm} (m \leqslant L)$ be the set of the real numbers $y_{nm} \in R(g_{nm})$ for which there exists a real number z' such that for an arbitrarily fixed z_{nm},

$$h_{nm}(y_{nm}, z') \leqslant z_{nm}. \tag{2.7}$$

Since $h_{nm}(y_{nm}, z')$ is nondecreasing and left-continuous in z', z_{n-1m} can be defined for fixed z_{nm} and fixed $y_{nm} \in V_{nm}$ as

$$z_{n-1m} = \max\{z'; h_{nm}(y_{nm}, z') \leqslant z_{nm}\}. \tag{2.8}$$

Define $G_{nm}(n = 1, 2,..., N)$ in a similar manner to (2.5) and (2.6). Then, for $n = 2, 3,..., N$,

$$\begin{aligned}
S_{nm}(z_{nm}) &\equiv \{(x_n, x_{n-1}, ..., x_1); G_{nm}(g_{nm}(x_n), g_{n-1m}(x_{n-1}), ..., \\
&\qquad g_{1m}(x_1)) \leqslant z_{nm} \text{ and } x_i \in X_i \ (i = 1, 2, ..., n)\} \\
&= \{(x_n, x_{n-1}, ..., x_1); h_{nm}(g_{nm}(x_n), G_{n-1m}(g_{n-1m}(x_{n-1}), ..., \\
&\qquad g_1(x_1))) \leqslant z_{nm} \text{ and } x_i \in X_i \ (i = 1, 2, ..., n)\} \\
&= \bigcup_{y_{nm} \in V_{nm}} \{\{x_n ; g_{nm}(x_n) = y_{nm} \text{ and } x_n \in X_n\} \times \{(x_{n-1}, ..., x_1); \\
&\qquad G_{n-1m}(g_{n-1m}(x_{n-1}), ..., g_{1m}(x_1)) \leqslant z_{n-1m} \text{ and } x_i \in X_i \\
&\qquad (i = 1, ..., n-1)\}\} \\
&= \bigcup_{y_{nm} \in V_{nm}} \{s_{nm}(y_{nm}) \times S_{n-1m}(z_{n-1m})\},
\end{aligned} \tag{2.9}$$

where

$$s_{nm}(y_{nm}) = \{x_n; g_{nm}(x_n) = y_{nm} \text{ and } x_n \in X_n\}. \tag{2.10}$$

Moreover, if $h_{nm}(y_{nm}, z_{n-1m})$ is also nondecreasing with respect to y_{nm}, then

$$S_{nm}(z_{nm}) = \bigcup_{y_{nm} \in V_{nm}} \{s'_{nm}(y_{nm}) \times S_{n-1m}(z_{n-1m})\}, \tag{2.11}$$

where

$$s'_{nm}(y_{nm}) = \{x_n ; g_{nm}(x_n) \leqslant y_{nm} \text{ and } x_n \in X_n\}. \tag{2.12}$$

Suppose that the principal equality constraint function $G_m(m \geqslant L + 1)$ is strictly decomposable with separating functions h_{nm} $(n = 2, 3,..., N)$. Let $V_{nm}(m \geqslant L + 1)$ be the set of the real numbers $y_{nm} \in R(g_{nm})$ for which there exists the real number z_{n-1m} such that for an arbitrary fixed z_{nm},

$$h_{nm}(y_{nm}, z_{n-1m}) = z_{nm} . \tag{2.13}$$

In a similar manner to (2.9), for $n = 2, 3,..., N$,

$$\begin{aligned}
S_{nm}(z_{nm}) &\equiv \{(x_n, x_{n-1}, ..., x_1); G_{nm}(g_{nm}(x_n), g_{n-1m}(x_{n-1}), ..., \\
&\qquad g_1(x_1)) = z_{nm} \text{ and } x_i \in X_i \ (i = 1, 2, ..., n)\} \\
&= \bigcup_{y_{nm} \in V_{nm}} \{s_{nm}(y_{nm}) \times S_{n-1m}(z_{n-1m})\},
\end{aligned} \tag{2.14}$$

where $s_{nm}(y_{nm})$ is defined by (2.10). Hence, if the K's principal constraint functions G_m(say, $m = 1, 2,..., K$) are strongly decomposable and left-continuous, the other principal constraint functions $G_m(m = K + 1,..., L)$ are weakly decomposable and left-continuous and the principal equality constraint functions $G_m(m = L + 1,..., M)$ are strictly decomposable, then, on account of (2.8) to (2.14), for $n = 2, 3,..., N$ and M-dimensional vectors $\mathbf{y}_n = (y_{n1}, y_{n2},..., y_{nM})$ and $\mathbf{z}_n = (z_{n1}, z_{n2},..., z_{nM})$,

$$\begin{aligned}
S_n(\mathbf{z}_n) &\equiv \{(x_n, x_{n-1},..., x_1);\ G_{nm}(g_{nm}(x_n), g_{n-1m}(x_{n-1}),..., \\
&\qquad g_{1m}(x_1)) \leqslant z_{nm}\ (m = 1, 2,..., L),\ G_{nm}(g_{nm}(x_n), g_{n-1m}(x_{n-1}),..., \\
&\qquad g_{1m}(x_1)) = z_{nm}\ (m = L + 1,..., M) \text{ and } x_i \in X_i\ (i = 1, 2,..., n)\} \\
&= \bigcap_{m=1}^{M} S_{nm}(z_{nm}) = \bigcup_{\mathbf{y}_n \in V_n} \left\{ \bigcap_{m=1}^{M} s_{nm}(y_{nm}) \times \bigcap_{m=1}^{M} S_{n-1m}(z_{n-1m}) \right\} \\
&= \bigcup_{\mathbf{y}_n \in V_n} \left\{ \left[\bigcap_{m=1}^{K} s'_{nm}(y_{nm}) \right] \times \left[\bigcap_{m=K+1}^{M} s_{nm}(y_{nm}) \right] \times \bigcap_{m=1}^{M} S_{n-1m}(z_{n-1m}) \right\},
\end{aligned} \tag{2.15}$$

where

$$V_n = \bigtimes_{m=1}^{M} V_{nm}\,. \tag{2.16}$$

Consequently,

$$\begin{aligned}
S_n(\mathbf{z}_n) &= \bigcup_{\mathbf{y}_n \in V_n} \{s_n(\mathbf{y}_n) \times S_{n-1}(\mathbf{z}_{n-1})\} \\
&= \bigcup_{\mathbf{y}_n \in V_n} \{s'_n(\mathbf{y}_n) \times S_{n-1}(\mathbf{z}_{n-1})\},
\end{aligned} \tag{2.17}$$

where

$$s_n(\mathbf{y}_n) = \{x_n\,;\ g_{nm}(x_n) = y_{nm}(m = 1, 2,..., M) \text{ and } x_n \in X_n\} \tag{2.18}$$

and

$$\begin{aligned}
s_n{}'(\mathbf{y}_n) = \{x_n\,;\ &g_{nm}(x_n) \leqslant y_{nm}\ (m = 1, 2,..., K),\ g_{nm}(x_n) = y_{nm} \\
&(m = K + 1,..., M) \text{ and } x_n \in X_n\}.
\end{aligned} \tag{2.19}$$

Suppose that the principal objective function is strongly decomposable with separating functions h_n $(n = 2, 3,..., N)$. Define F_n $(n = 1, 2,..., N)$ in a similar manner to (2.5) and (2.6). Furthermore, define $P_n(\mathbf{z}_n)$ $(n = 1, 2,..., N)$ and $p_n(\mathbf{y}_n)$ $(n = 2, 3,..., N)$ as follows:

$$P_n(\mathbf{z}_n) = \mathrm{OPT}\{F_n(f_n(x_n), f_{n-1}(x_{n-1}),..., f_1(x_1)) \mid (x_n, x_{n-1},..., x_1) \in S_n(\mathbf{z}_n)\} \tag{2.20}$$

and

$$p_n(\mathbf{y}_n) = \text{OPT}\{f_n(x_n) \mid x_n \in s_n'(\mathbf{y}_n)\}. \tag{2.21}$$

THEOREM. *Suppose that the principal objective function is strongly decomposable with separating function h_n $(n = 2, 3,..., N)$, for $m = 1, 2,..., K$ $(m = K + 1,..., L)$ the principal constraint functions G_m are strongly (weakly) decomposable and left-continuous with separating function h_{nm} $(n = 2, 3,..., N)$ and the principal equality constraint functions are strictly decomposable with separating functions h_{nm} $(n = 2, 3,..., N)$. Then the principal problem can be decomposed into subproblems and the following recursive relations hold for $n = 2, 3,..., N$:*

$$P_n(\mathbf{z}_n) = \text{OPT}\{h_n(p_n(\mathbf{y}_n), P_{n-1}(\mathbf{z}_{n-1})) \mid \mathbf{y}_n \in V_n\}. \tag{2.22}$$

Proof. The proof is straightforward. Since $h_n(y, z)$ is nondecreasing with respect to both y and z, it follows from (2.15), (2.16), (2.17) and (2.19) that

$$\begin{aligned}
P_n(\mathbf{z}_n) &= \text{OPT}\{h_n(f_n(x_n), F_{n-1}(f_{n-1}(x_{n-1}),..., f_1(x_1))) \mid (x_n, x_{n-1},..., x_1) \\
&\qquad \in \bigcup_{\mathbf{y}_n \in V_n} \{s_n'(\mathbf{y}_n) \times S_{n-1}(\mathbf{z}_{n-1})\}\} \\
&= \text{OPT}\{\text{OPT}[h_n(f_n(x_n), F_{n-1}(f_{n-1}(x_{n-1}),..., f_1(x_1))) \mid (x_n, x_{n-1},..., x_1) \\
&\qquad \in \{s_n'(\mathbf{y}_n) \times S_{n-1}(\mathbf{z}_{n-1})\}] \mid \mathbf{y}_n \in V_n\} \\
&= \text{OPT}\{h_n(\text{OPT}[f_n(x_n) \mid x_n \in s_n'(\mathbf{y}_n)], \text{OPT}[F_{n-1}(f_{n-1}(x_{n-1}),..., \\
&\qquad f_1(x_1)) \mid (x_{n-1},..., x_1) \in S_{n-1}(\mathbf{z}_{n-1})]) \mid \mathbf{y}_n \in V_n\} \\
&= \text{OPT}\{h_n(p_n(\mathbf{y}_n), P_{n-1}(\mathbf{z}_{n-1})) \mid \mathbf{y}_n \in V_n\}.
\end{aligned}$$

The above theorem shows that under the strong decomposability of the principal objective function, the weak or strong decomposability and the left-continuity of the principal constraint functions and the strict decomposability of the principal equality constraint functions, the principal problem can be decomposed into subproblems $P_1(\mathbf{z}_1), p_2(\mathbf{y}_2),..., p_N(y_N)$ by dynamic programming. Therefore, the principal problem $P \equiv P_N(\mathbf{0})$ can be solved by composing $P_n(\mathbf{z}_n)$ by means of the recursive relation (2.22) and the solutions and the optimum values of the subproblems $p_n(\mathbf{y}_n)$. This sequence is represented graphically in Fig. 1. It is to be noted that the inequality constraints $\{g_{nm}(x_n) \leqslant y_{nm}$ $(m = 1, 2,..., K)\}$ of the subproblems $p_n(\mathbf{y}_n)$, in general, should be replaced by the equality constraints $\{g_{nm}(x_n) = y_{nm}$ $(m = 1, 2,..., K)\}$. That is, in general, $s_n'(\mathbf{y}_n)$ should be replaced by $s_n(\mathbf{y}_n)$. Because $s_n(\mathbf{y}_n)$ is more restrictive than $s_n'(\mathbf{y}_n)$. However, when the principal problem is solved on a digital computer, the inequality constraints $s_n'(\mathbf{y}_n)$ may give more optimal value than the equality constraints $s_n(\mathbf{y}_n)$ does,

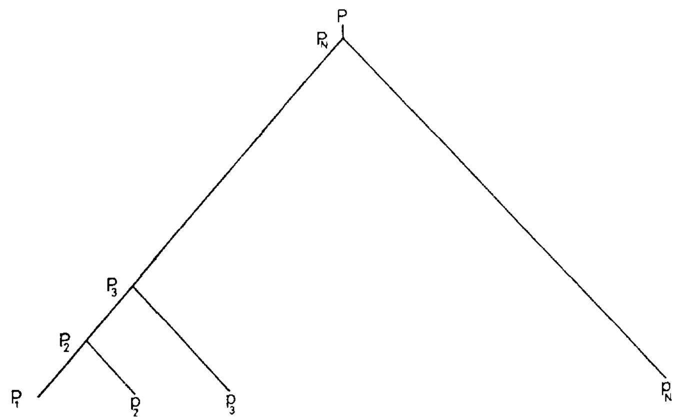

FIGURE 1

because the calculation of the subproblems $p_n(\mathbf{y}_n)$ is done at the finite grid of $\mathbf{y}_n$. Moreover, in particular but important cases, inequality constraints $s_n'(\mathbf{y}_n)$ are more convenient than the equality constraints $s_n(\mathbf{y}_n)$; for example, the case that subproblems are equivalent to linear programming problems.

Suppose that for $1 < N_1 < \cdots < N_{r-1} < N_r = N$,

$$F(f_N(x_N),\ldots, f_1(x_1)) = H[F^r(f_N(x_N),\ldots, f_{N_{r-1}+1}(x_{N_{r-1}+1})),\ldots,$$
$$F^1(f_{N_1}(x_{N_1}),\ldots, f_1(x_1))]$$

and

$$G_m(g_{Nm}(x_N),\ldots, g_{1m}(x_1)) = H_m[G_m{}^r(g_{Nm}(x_N),\ldots, g_{N_{r-1}+1m}(x_{N_{r-1}+1})),\ldots,$$
$$G_m{}^1(g_{N_1m}(x_{N_1}),\ldots, g_{1m}(x_1))], \qquad m = 1, 2,\ldots, M,$$

where H is strongly decomposable, H_m $(m = 1, 2,\ldots, L)$ are strongly decomposable and left-continuous and H_m $(m = L + 1,\ldots, M)$ are strictly decomposable. If F^i and $G_m{}^i$ $(i = 1, 2,\ldots, r)$ satisfy the conditions of the theorem, then the principal problem can be decomposed into subproblems by dynamic programming. The above result is a generalization of parallel operations [2 (p. 35), 9]. This decomposition is represented graphically in Fig. 2.

The proof of the theorem suggests the sufficient condition for the subproblem to be decomposed by dynamic programming. Consider the general problem:

$$\mathrm{OPT}\, f(x_N, x_{N-1},\ldots, x_1)$$

subject to

$$g_m(x_N, x_{N-1},\ldots, x_1) \leqslant 0, \qquad m = 1, 2,\ldots, L,$$

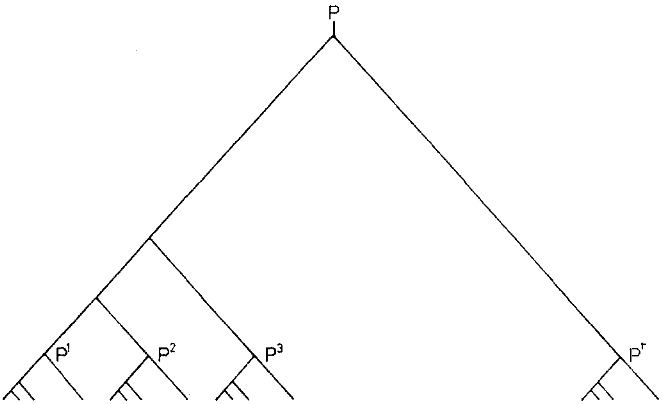

FIGURE 2

and

$$g_m(x_N, x_{N-1}, \ldots, x_1) = 0, \qquad m = L+1, \ldots, M,$$

where x_n $(n = 1, 2, \ldots, N)$ are real variables (this notation may cause no confusion) and f, g_m $(m = 1, 2, \ldots, M)$ are real valued functions. The subproblems are given by this general form, respectively. Moreover, this type of problem is also the generalized form of the principal problem. Suppose that f, g_m $(m = 1, 2, \ldots, M)$ are separable. Define f_n, g_{nm} and V_{nm} in a similar manner to (2.5), (2.6), (2.7) and (2.13).

COROLLARY. *Suppose that the objective function f is weakly decomposable with separating functions h_n, the inequality constraint functions g_m $(m \leqslant L)$ are weakly decomposable and left-continuous with separating functions h_{nm} and the equality constraint functions g_m $(m > L)$ are strictly decomposable with separating functions h_{nm}. Then, for $n = 2, 3, \ldots, N$ and real vectors $\mathbf{z}_n = (z_{n1}, \ldots, z_{nM})$,*

$$P_n(\mathbf{z}_n) = \mathrm{OPT}\{h_n(x_n, P_{n-1}(\mathbf{z}_{n-1})) \mid x_n \in V_n\},$$

where

$$P_n(\mathbf{z}_n) = \mathrm{OPT}\{f_n(x_n, x_{n-1}, \ldots, x_1) \mid (x_n, x_{n-1}, \ldots, x_1) \in S_n(\mathbf{z}_n)\},$$

$$S_n(\mathbf{z}_n) = \{(x_n, x_{n-1}, \ldots, x_1); g_{nm}(x_n, x_{n-1}, \ldots, x_1) \leqslant z_{nm}$$
$$(m = 1, 2, \ldots, L), g_{nm}(x_n, x_{n-1}, \ldots, x_1) = z_{nm} \ (m = L+1, \ldots, M)\},$$

$$V_n = \mathop{\times}_{m=1}^{M} V_{nm}$$

and $\mathbf{z}_{n-1}$ is the vector determined uniquely by x_n and $\mathbf{z}_n$.

The proof is omitted, because it is quite similar to the proof of the theorem. Hence, if all subproblems $p_n(\mathbf{y}_n)$ satisfy the conditions of the corollary, then the principal problem can be solved by "multi-level" dynamic programming.

In the above, the weak, strong or strict decomposability has been defined as the nondecrease or increase of all separating functions. These definitions, however, can be generalized as the monotonicity or strict monotonicity of all separating functions. That is, the real valued function H is said to be *weakly* (*strongly*) *decomposable*, if H is separable and all separating functions $h_n(y, z)$ are monotone with respect to z (both y and z). Moreover, H is said to be *strictly decomposable*, if H is separable and all separating functions $h_n(y, z)$ are strictly monotone with respect to z. Under these generalized definitions, the theorem and the corollary are valid with some modification. For example, suppose that the separating functions h_n and h_{nm} ($n = 2, 3,..., N$, $m = 1, 2,..., L$) in the theorem are nonincreasing. Moreover, suppose that $h_{nm}(y, z)$ are right-continuous in z. Let $\overline{\text{OPT}} = \min$ or max, according as OPT $=$ max or min. Then the theorem is valid with the following recursive relations in stead of (2.22):

$$P_n(\mathbf{z}_n) = \text{OPT}\{h_n(\bar{p}_n(\mathbf{y}_n), \bar{P}_{n-1}(\mathbf{z}_{n-1})) \mid \mathbf{y}_n \in V_n\},$$

where

$$\bar{p}_n(\mathbf{y}_n) = \overline{\text{OPT}}\{f_n(x_n) \mid x_n \in s_n'(\mathbf{y}_n)\},$$

$$\bar{P}_{n-1}(\mathbf{z}_{n-1}) = \overline{\text{OPT}}\{F_{n-1}(f_{n-1}(x_{n-1}),..., f_1(x_1)) \mid (x_{n-1},..., x_1) \in S_{n-1}(\mathbf{z}_{n-1})\}$$

and V_n, $s_n'(\mathbf{y}_n)$ and $S_{n-1}(\mathbf{z}_{n-1})$ are defined with $\leqslant$ substituted by $\geqslant$. The similar results hold also for the other situations. Finally, examples of the separating function $h(y, z)$ which is monotone with respect to both y and z are: for real variables y and z, $h(y, z) = y + z$, $= y - z$, $= \text{OPT}\{y, z\}$ and so on; for positive or negative variables y and z, $h(y, z) = yz$, $= y/z$, $= z/y$ and so on; for nonnegative y, z, a, b, c and d, e, f, $g \geqslant 1$, $h(y, z) = ay^d + by^e z^f + cz^g$. Strongly decomposable functions are composed of them and are fairly general functions. Weakly or strictly decomposable functions are more general than strongly decomposable ones.

3. Application to Block-Diagonal Geometric Program

The principal problem whose subproblems are equivalent to linear programming problems is discussed in this section. That is, the objective function and constraint functions of the subproblem are expressed by

$$f_n(x_n) = \sum_{i=1}^{i_n} c_{ni} x_{ni} + c_n \tag{3.1}$$

and

$$g_{nm}(x_n) = \sum_{i=1}^{i_n} a_{nmi} x_{ni} + b_{nm}, \qquad m = 1, 2, ..., M. \tag{3.2}$$

The following type of subproblem also can be solved by linear programming:

$$f_n(x_n) = c_n \prod_{i=1}^{i_n} x_{ni}{}^{a_{ni}} \tag{3.3}$$

and

$$g_{nm}(x_n) = c_{nm} \prod_{i=1}^{i_n} x_{ni}{}^{a_{nmi}}, \qquad m = 1, 2, ..., M, \tag{3.4}$$

where $x_{ni} > 0$. In the sequel, the principal equality constraints are omitted. Weakly or strongly decomposable functions are fairly general functions, as mentioned in the preceding section. Hence, there are many variety of the principal problems decomposed into linear subproblems. For example of the principal objective function,

$$\begin{aligned}
F &= \mathrm{OPT}\{f_N, f_{N-1}, ..., f_1\}, \\
F &= \Sigma f_n + \Pi f_{n'}, \\
F &= \Sigma \Pi f_n, \\
F &= \Pi \Sigma f_n, \\
F &= \Sigma f_n / \Sigma f_{n'}, \\
F &= \Pi f_n / \Pi f_{n'}
\end{aligned}$$

and so on. The most interesting and simple problem may be the following:

$$\min \sum_{n=1}^{N} f_n(x_n)$$

subject to

$$\sum_{n=1}^{N} g_{nm}(x_n) \leqslant 1, \qquad m = 1, 2, ..., M,$$

$$g_{nm}(x_n) \leqslant 1, \qquad m = M+1, ..., M_n, \qquad n = 1, 2, ..., N,$$

and

$$x_n > 0, \qquad n = 1, 2, ..., N,$$

where f_n, g_{nm} are given in (3.3) and (3.4), respectively. The above problem is a block-diagonal form of generalized polynomial optimization problem [17].

If f_n, g_{nm} are given in (3.1) and (3.2) respectively, the above problem is a block-diagonal linear programming problem and was discussed by Dantzig and Wolfe [5] and Nemhauser [14]. In the following, c_n and c_{nm} are assumed to be positive. Then, the above problem is the *block-diagonal* geometric program. The principal objective function and the principal constraint functions are strongly decomposable and left-continuous. Moreover, $f_n(x_n), g_{nm}(x_n) > 0$. Hence, it follows from the theorem that for $n = 1, 2,..., N$ and $0 < z_{nm} \leqslant 1 (m = 1, 2,.... M)$,

$$P_n(\mathbf{z}_n) = \min\{p_n(\mathbf{y}_n) + P_{n-1}(\mathbf{z}_n - \mathbf{y}_n) \mid \mathbf{y}_n \in V_n\}, \tag{3.5}$$

where

$$P_n(\mathbf{z}_n) = \min\left\{\sum_{i=1}^{n} f_i(x_i) \,\middle|\, \sum_{i=1}^{n} g_{im}(x_i) \leqslant z_{nm}\ (m = 1, 2,..., M) \text{ and } x_i \in X_i\ (i = 1, 2,..., n)\right\}, \tag{3.6}$$

$$P_0(\mathbf{z}_0) = 0 \qquad \text{for arbitrary } \mathbf{z}_0, \tag{3.7}$$

$$\begin{aligned} p_n(\mathbf{y}_n) &= \min\{f_n(x_n) \mid g_{nm}(x_n) \leqslant y_{nm}\ (m = 1, 2,..., M) \text{ and } x_n \in X_n\}, \\ X_n &= \{x_n\,;\, g_{nm}(x_n) \leqslant 1\ (m = M + 1,..., M_n) \text{ and } x_n > 0\}, \end{aligned} \tag{3.8}$$

and

$$V_n = \mathop{\times}_{m=1}^{M} (0, z_{nm}).$$

Under the transformations $t_{ni} = \log x_{ni}$, $b_n = \log c_n$ and $b_{nm} = \log c_{nm}$, the subproblems $p_n(\mathbf{y}_n)$ are equivalent to the following:

$$\min \sum_{i=1}^{i_n} a_{ni} t_{ni} + b_n$$

subject to

$$\sum_{i=1}^{i_n} a_{nmi} t_{ni} + b_{nm} \leqslant \log y_{nm}, \qquad m = 1, 2,..., M,$$

and

$$\sum_{i=1}^{i_n} a_{nmi} t_{ni} + b_{nm} \leqslant 0, \qquad m = M + 1,..., M_n.$$

Consequently, the subproblems can be solved by parametric linear programming with free variables. Hence, the block-diagonal geometric program can be solved by composing their solutions and minimal values by dynamic programming. In case of few principal constraints, this method is of great utility, whether the degree of difficulty is zero or positive. Clasen [4] devised a computer code for solving the dual geometric program. In the sequel, consider the block-diagonal geometric program with one principal constraint. If it has some principal constraints, then a number of techniques which reduce computation time or high-speed memory requirement [2, 15] may be effective. Let $v_n(y_n)$ be the solution $(x_{n1}, x_{n2}, \ldots, x_{ni_n})$ of the subproblems

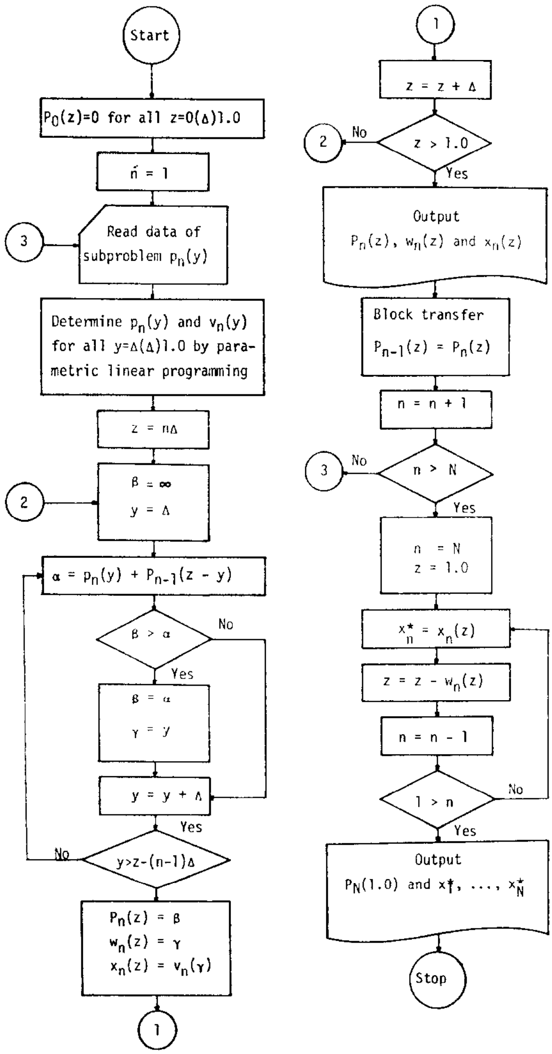

FIGURE 3

$p_n(y_n)$. Let $w_n(z_n)$ be the allocation y_n that minimizes $P_n(z_n)$ in the recursive relation (3.5). Let $x_n(z_n)$ be the point $(x_{n1}, x_{n2}, ..., x_{ni_n})$ that minimizes $P_n(z_n)$. That is, $x_n(z_n) = v_n(w_n(z_n))$. Let the grid size be Δ. $P_n(z_n)$ and $p_n(y_n)$ are evaluated over the partition $\Delta(\Delta)1.0$, which is read "from Δ to 1.0 in steps of Δ". Let $(x_1^*, x_2^*, ..., x_N^*)$ be the solution of the block-diagonal geometric program. The minimal value is $P_N(1.0)$. The flow chart for solving the block-diagonal geometric program is given in Fig. 3. Consider the concrete example:

$$\min 2.0x_{11}^{0.9}x_{12}^{-1.5}x_{13}^{-3.0} + 5.0x_{21}^{-0.3}x_{22}^{2.6} + 4.7x_{31}^{-1.8}x_{32}^{-0.5}x_{33}^{1.0}$$

subject to

$$7.2x_{11}^{-3.8}x_{12}^{2.2}x_{13}^{4.3} + 0.5x_{21}^{-0.7}x_{22}^{-1.6} + 0.2x_{31}^{4.3}x_{32}^{-1.9}x_{33}^{8.5} \leqslant 1,$$

$$10.0x_{11}^{2.3}x_{12}^{1.7}x_{13}^{4.5} \leqslant 1, \qquad 0.2x_{21}^{-2.1}x_{22}^{0.4} \leqslant 1, \qquad 6.2x_{31}^{4.2}x_{32}^{-2.7}x_{33}^{-0.6} \leqslant 1,$$

$$3.1x_{11}^{1.6}x_{12}^{0.4}x_{13}^{-3.8} \leqslant 1, \qquad 3.7x_{21}^{5.4}x_{22}^{1.3} \leqslant 1, \qquad 0.3x_{31}^{-1.1}x_{32}^{7.3}x_{33}^{-5.6} \leqslant 1,$$

and

$$x_{ni} > 0.$$

Let Δ be 0.01. The minimal value and the solution $(x_{11}^*, x_{12}^*, x_{13}^*, x_{21}^*, x_{22}^*, x_{31}^*, x_{32}^*, x_{33}^*)$ of the above example are 19.13042 and (0.86825, 0.24585, 1.09479, 1.05807, 0.28913, 0.75083, 1.22834, 1.11561), respectively. The sequences $P_n(z)$, $w_n(z)$ and $x_n(z)$ $(n = 1, 2, 3)$ over the partition 0.50 (0.01) 1.0 of z are given in Table 1. The computation time is 16 sec on the FACOM 230-60 which nearly corresponds to an IBM 360-60. In order to test the present method, consider the block-diagonal geometric program with difficulty zero, which can be solved immediately by the duality theory of geometric programming [6]. It is to be noted, however, that the block-diagonal geometric program with difficulty zero includes at least one linear subproblem with an unbounded solution. Because $t_{ni} = \log x_{ni}$ are free variables (that is, variables without fixed signs). The illustrative example is:

$$\min x_{11}^{0.2}x_{12}^{-1.0} + 2.8x_{21}^{2.5}x_{22}^{1.0}x_{23}^{-2.0}$$

subject to

$$3.0x_{11}^{-0.9}x_{12}^{2.6} + 1.7x_{21}^{-3.0}x_{22}^{0.6}x_{23}^{-1.2} \leqslant 1,$$

$$0.8x_{11}^{1.5}x_{12}^{1.4} \leqslant 1, \qquad 0.5x_{21}^{2.4}x_{22}^{-2.1}x_{23}^{0.4} \leqslant 1,$$

and

$$x_{ni} > 0.$$

Since the above problem is superconsistent, it follows from the duality theory of geometric programming that the minimal value and the solution of the

Table I.

z	$P_1(z)$	$w_1(z)$	$x_{11}(z)$	$x_{12}(z)$	$x_{13}(z)$	$P_2(z)$	$w_2(z)$	$x_{21}(z)$	$x_{22}(z)$	$P_3(z)$	$w_3(z)$	$x_{31}(z)$	$x_{32}(z)$	$x_{33}(z)$
0.50	13.77440	0.50	0.93241	0.21425	1.11193	14.79660	0.05	0.99986	0.36574	23.86250	0.04	0.70331	1.13269	1.01669
0.51	13.65440	0.51	0.92982	0.21540	1.11126	14.65714	0.05	0.99986	0.36574	23.70324	0.04	0.70331	1.13269	1.01669
0.52	13.53774	0.52	0.92728	0.21654	1.11060	14.52197	0.05	0.99986	0.36574	23.54936	0.04	0.70331	1.13269	1.01669
0.53	13.42428	0.53	0.92480	0.21766	1.10995	14.39089	0.05	0.99986	0.36574	23.40056	0.04	0.70331	1.13269	1.01669
0.54	13.31386	0.54	0.92237	0.21877	1.10931	14.26370	0.05	0.99986	0.36574	23.25655	0.04	0.70331	1.13269	1.01669
0.55	13.20636	0.55	0.91999	0.21986	1.10869	14.14019	0.05	0.99986	0.36574	23.11708	0.04	0.70331	1.13269	1.01669
0.56	13.10163	0.56	0.91766	0.22094	1.10808	14.02019	0.05	0.99986	0.36574	22.98192	0.04	0.70331	1.13269	1.01669
0.57	12.99957	0.57	0.91538	0.22200	1.10748	13.90353	0.05	0.99986	0.36574	22.84828	0.05	0.71460	1.15527	1.03994
0.58	12.90005	0.58	0.91314	0.22305	1.10688	13.79007	0.05	0.99986	0.36574	22.71720	0.05	0.71460	1.15527	1.03994
0.59	12.80298	0.59	0.91095	0.22409	1.10630	13.67965	0.05	0.99986	0.36574	22.59000	0.05	0.71460	1.15527	1.03994
0.60	12.70825	0.60	0.90880	0.22512	1.10573	13.57214	0.05	0.99986	0.36574	22.46649	0.05	0.71460	1.15527	1.03994
0.61	12.61578	0.61	0.90669	0.22613	1.10517	13.46662	0.06	1.03100	0.32200	22.34649	0.05	0.71460	1.15527	1.03994
0.62	12.52546	0.62	0.90462	0.22713	1.10462	13.36190	0.06	1.03100	0.32200	22.22984	0.05	0.71460	1.15527	1.03994
0.63	12.43722	0.63	0.90258	0.22812	1.10408	13.25983	0.06	1.03100	0.32200	22.11638	0.05	0.71460	1.15527	1.03994
0.64	12.35098	0.64	0.90059	0.22910	1.10355	13.16032	0.06	1.03100	0.32200	22.00596	0.05	0.71460	1.15527	1.03994
0.65	12.26665	0.65	0.89862	0.23006	1.10302	13.06325	0.06	1.03100	0.32200	21.89834	0.06	0.72396	1.17406	1.05933
0.66	12.18418	0.66	0.89670	0.23102	1.10251	12.96852	0.06	1.03100	0.32200	21.79083	0.06	0.72396	1.17406	1.05933
0.67	12.10349	0.67	0.89480	0.23196	1.10200	12.87604	0.06	1.03100	0.32200	21.68531	0.06	0.72396	1.17406	1.05933
0.68	12.02452	0.68	0.89294	0.23290	1.10150	12.78573	0.06	1.03100	0.32200	21.58058	0.06	0.72396	1.17406	1.05933
0.69	11.94720	0.69	0.89111	0.23382	1.10101	12.69749	0.06	1.03100	0.32200	21.47852	0.06	0.72396	1.17406	1.05933
0.70	11.87149	0.70	0.88930	0.23474	1.10052	12.61124	0.06	1.03100	0.32200	21.37900	0.06	0.72396	1.17406	1.05933
0.71	11.79731	0.71	0.88753	0.23564	1.10004	12.52692	0.06	1.03100	0.32200	21.28193	0.06	0.72396	1.17406	1.05933
0.72	11.72463	0.72	0.88579	0.23654	1.09957	12.44445	0.06	1.03100	0.32200	21.18720	0.06	0.72396	1.17406	1.05933
0.73	11.65339	0.73	0.88407	0.23743	1.09911	12.36376	0.06	1.03100	0.32200	21.09473	0.06	0.72396	1.17406	1.05933
0.74	11.58354	0.74	0.88238	0.23831	1.09865	12.28478	0.06	1.03100	0.32200	21.00441	0.06	0.72396	1.17406	1.05933
0.75	11.51505	0.75	0.88071	0.23918	1.09820	12.20747	0.06	1.03100	0.32200	20.91450	0.07	0.73196	1.19018	1.07601
0.76	11.44785	0.76	0.87907	0.24004	1.09775	12.13175	0.06	1.03100	0.32200	20.82626	0.07	0.73196	1.19018	1.07601
0.77	11.38192	0.77	0.87746	0.24089	1.09731	12.05758	0.06	1.03100	0.32200	20.74002	0.07	0.73196	1.19018	1.07601
0.78	11.31721	0.78	0.87587	0.24174	1.09688	11.98490	0.06	1.03100	0.32200	20.65569	0.07	0.73196	1.19018	1.07601
0.79	11.25369	0.79	0.87430	0.24257	1.09645	11.91366	0.06	1.03100	0.32200	20.57322	0.07	0.73196	1.19018	1.07601
0.80	11.19131	0.80	0.87275	0.24340	1.09603	11.84381	0.06	1.03100	0.32200	20.49253	0.07	0.73196	1.19018	1.07601
0.81	11.13005	0.81	0.87123	0.24423	1.09561	11.77531	0.06	1.03100	0.32200	20.41356	0.07	0.73196	1.19018	1.07601
0.82	11.06987	0.82	0.86973	0.24504	1.09520	11.70812	0.06	1.03100	0.32200	20.33625	0.07	0.73196	1.19018	1.07601
0.83	11.01074	0.83	0.86825	0.24585	1.09479	11.64219	0.06	1.03100	0.32200	20.25916	0.08	0.73897	1.20433	1.09067
0.84	10.95263	0.84	0.86678	0.24665	1.09439	11.57710	0.07	1.05807	0.28913	20.18344	0.08	0.73897	1.20433	1.09067
0.85	10.89551	0.85	0.86534	0.24744	1.09399	11.51239	0.07	1.05807	0.28913	20.10927	0.08	0.73897	1.20433	1.09067
0.86	10.83934	0.86	0.86392	0.24823	1.09360	11.44887	0.07	1.05807	0.28913	20.03659	0.08	0.73897	1.20433	1.09067
0.87	10.78412	0.87	0.86252	0.24901	1.09321	11.38650	0.07	1.05807	0.28913	19.96535	0.08	0.73897	1.20433	1.09067
0.88	10.72979	0.88	0.86113	0.24978	1.09283	11.32524	0.07	1.05807	0.28913	19.89550	0.08	0.73897	1.20433	1.09067
0.89	10.67636	0.89	0.85976	0.25055	1.09245	11.26506	0.07	1.05807	0.28913	19.82700	0.08	0.73897	1.20433	1.09067
0.90	10.62378	0.90	0.85841	0.25131	1.09208	11.20593	0.07	1.05807	0.28913	19.75961	0.09	0.74521	1.21695	1.10376
0.91	10.57203	0.91	0.85708	0.25207	1.09171	11.14781	0.07	1.05807	0.28913	19.69242	0.09	0.74521	1.21695	1.10376
0.92	10.52110	0.92	0.85576	0.25282	1.09134	11.09069	0.07	1.05807	0.28913	19.62649	0.09	0.74521	1.21695	1.10376
0.93	10.47097	0.93	0.85446	0.25356	1.09098	11.03453	0.07	1.05807	0.28913	19.56140	0.09	0.74521	1.21695	1.10376
0.94	10.42160	0.94	0.85318	0.25430	1.09062	10.97930	0.07	1.05807	0.28913	19.49670	0.09	0.74521	1.21695	1.10376
0.95	10.37298	0.95	0.85191	0.25503	1.09027	10.92498	0.07	1.05807	0.28913	19.43317	0.09	0.74521	1.21695	1.10376
0.96	10.32510	0.96	0.85065	0.25575	1.08992	10.87154	0.07	1.05807	0.28913	19.37080	0.09	0.74521	1.21695	1.10376
0.97	10.27793	0.97	0.84941	0.25647	1.08957	10.81896	0.07	1.05807	0.28913	19.30954	0.09	0.74521	1.21695	1.10376
0.98	10.23146	0.98	0.84819	0.25719	1.08923	10.76722	0.07	1.05807	0.28913	19.24936	0.09	0.74521	1.21695	1.10376
0.99	10.18567	0.99	0.84698	0.25790	1.08889	10.71629	0.07	1.05807	0.28913	19.18955	0.10	0.75083	1.22834	1.11561
1.00	10.14053	1.00	0.84578	0.25860	1.08855	10.66615	0.07	1.05807	0.28913	19.13042	0.10	0.75083	1.22834	1.11561

above problem is 1.43698 and $x_{11}^* = 1.5076$, $x_{12}^* = 0.75545$. On the other hand, the present method gives the minimal value 1.44197 and the solution $x_{11} = 1.5117$, $x_{12} = 0.75324$. The difference between two solutions results from the fact $y_2 \geqslant 0.01$. If y_2 is allowed to be zero, then the present method gives the same solution as geometric programming. Moreover, the simplex tableau of the second linear subproblem shows that

$$x_{21}^* = \exp\{-0.72770 - 0.71428u\},$$

$x_{22}^* = 1.0$ and $x_{23}^* = \exp\{6.0991 + 1.7858u\}$, where $u \to \infty$. The computation time is 8 sec on the FACOM 230-60.

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References

1. R. E. Bellman, "Dynamic Programming," Princeton Univ. Press, Princeton, N. J., 1957.
2. R. E. Bellman and S. E. Dreyfus, "Applied Dynamic Programming," Princeton Univ. Press, Princeton, N. J., 1962.
3. F. Bessiere and E. A. Sautter, Optimization and suboptimization: the method of extended models in the non-linear case, *Management Sci.* **15** (1968), 1–11.
4. R. J. Clasen, "The Numerical Solution of the Chemical Equilibrium Problem," RAND Corp. Memo RM-4345-PR, 1965.
5. G. B. Dantzig and P. Wolfe, The decomposition principle for linear programming, *Operations Res.* **8** (1960), 101–111.
6. R. J. Duffin, E. L. Peterson, and C. Zener, "Geometric Programming," John Wiley, New York, N. Y., 1967.
7. J. E. Falk and R. M. Soland, An algorithm for separable nonconvex programming problems, *Management Sci.* **15** (1969), 550–569.
8. W. Findeisen, "Parametric optimization by primal method in multilevel systems," IEEE SSC-**4** (1968), 155–164.
9 P. A. Gilmore, Structuring of parallel algorithms, *J. Assoc. Comput. Mach.* **15** (1968), 176–192.
10. A. Kaufmann and R. Cruon, "Dynamic Programming," Academic Press, New York, 1967.
11. L. S. Lasdon, "Duality and decomposition in mathematical programming," IEEE SSC-**4** (1968), 86–100.
12. L. G. Mitten, Composition principle for synthesis of optimal multistage processes, *Operations Res.* **12** (1964), 610–619.
13. P. V. Moeseke and G. D. Ghellinck, Decentralization in separable programming, *Econometrica* **37** (1969), 73–78.
14. G. L. Nemhauser, Decomposition of linear programs by dynamic programming, *Naval Res. Logist. Quart.* **11** (1964), 191–195.

15. G. L. NEMHAUSER, "Introduction to Dynamic Programming," John Wiley, New York, 1966.
16. G. E. NEUNER AND R. N. MILLER, "Effectiveness Evaluation Using Dynamic Programming," Proceedings 1968 Annual Symposium on Reliability, Boston, 47–61.
17. U. PASSY AND D. J. WILDE, Generalized polynomial optimization, *SIAM J. Appl. Math.* **15** (1967), 1344–1356.
18. P. VARAIYA, Decomposition of large-scale systems, *SIAM J. Control* **4** (1966), 173–178.